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Recursive Relation for Zero Inflated Poisson Mixture Distributions

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Abstract

The paper extends the work of Sarguta who derived recursive relations for univariate distributions by considering the ZIP continuous mixtures. The paper gives a recursive formular which can be used to evaluate the mixed distributions which can be used when the probability distribution functions cannot be evaluated explicitly. Integration by parts is often employed when deriving the recursive formulas. From section two up to section seven, we derived the recursive formulas for ZIP mixture distributions using Rectangular, Exponential, Gamma with two parameters, Poisson- Beta and Inverted - Beta as mixing distributions.

Keywords: ZIP, recursive, inflated model, prior distributions and integration

1 Introduction

A zero-inflated model is a statistical model based on a zero-inflated probability distribution. Gardner et.al in their paper, [9], suggested that using an inflation technique was adequate if the intention is to estimate the effect of the covariates. In April, 2009, the University of Carlos the third team in their [10], assessed the impacts of the fertility decisions of mothers on infant mortality. They used a Poisson regression to model the number of children. They fitted an inflated zero's model with negative binomial to the fertility decisions so as to eliminate the problem of overdispersion of the Poisson model. A main difficulty with the use of Mixed Poisson distribution is that, with the exception of a few mixing distributions, their probability mass function $f(x)$ is difficult to evaluate [8]. One way of circumventing this problem is to express the mixed distributions in terms of recursive relations. A number of methods for deriving such recursive relations have been developed, starting with the works of [5], [4], [1], [3], etc. Integration by parts does not require assumptions given by [3] or by [1].

2 Rectangular mixing distribution

Therefore, the mixed distribution becomes

$$
\begin{aligned}
Prob(Y=k) &= \begin{cases} \int_a^b [\rho + (1-\rho)e^{-\lambda}]\frac{1}{b-a} d\lambda, & \text{k=0;} \\ \int_0^\infty [(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}]\frac{1}{b-a} d\lambda, & \text{k=}1,2,\ldots. \end{cases} \\
&= \begin{cases} \rho + \frac{(1-\rho)}{b-a}[\int_0^b e^{-\lambda} - \int_0^a e^{-\lambda}]d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!(b-a)}[\int_0^b e^{-\lambda}\lambda^k d\lambda - \int_0^a e^{-\lambda}\lambda^k d\lambda], & \text{k=}1,2,\ldots. \end{cases} \\
&= \begin{cases} \rho + \frac{(1-\rho)}{b-a}\left[e^{-a} - e^{-b}\right], & \text{k=0;} \\ \frac{(1-\rho)}{k!(b-a)}\left[\Gamma_b(k+1) - \Gamma_a(k+1)\right], & \text{k=}1,2,\ldots. \end{cases}
\end{aligned}
$$

Let us consider the following function

$$\int_0^b e^{-\lambda}\lambda^k d\lambda$$

, using integration by parts

$$
\begin{aligned}
\int_0^b e^{-\lambda}\lambda^k d\lambda &= -\lambda^k e^{-\lambda}|_0^b + k\int_0^b e^{-\lambda}\lambda^{k-1} d\lambda && (1) \\
&= -[b^k e^{-b} - 0] + k\Gamma_b k && (2)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Gamma_b(k+1) &= -b^k e^{-b} + k\Gamma_b k \\
&= -b^k e^{-b} - kb^{k-1}e^{-b} + k(k-1)\Gamma_b(k-1) \\
&= -b^k e^{-b} - kb^{k-1}e^{-b} + k(k-1)[-b^{k-2}e^{-b} + (k-2)\Gamma_b(k-2)] \\
&= -e^{-b}[b^k + kb^{k-1} + k(k-1)b^{k-2} + k(k-1)(k-2)b^{k-3} + \cdots \\
&\quad + k(k-1)(k-2)(k-3)\cdots[k-(k-1)]b^{k-k}]
\end{aligned}
$$

Therefore

$$\frac{(1-\rho)}{k!(b-a)}\Gamma_b(k-1)=(1-\rho)\frac{-e^{-b}}{(b-a)}\left[\frac{b^k}{k!}+\frac{b^{k-1}}{(k-1)!}+\frac{b^{k-2}}{(k-2)!}+\cdots+\frac{b}{1!}+\frac{1}{0!}\right]$$

Similarly

$$\frac{(1-\rho)}{k!(b-a)}\Gamma_a(k-1)=(1-\rho)\frac{-e^{-a}}{(b-a)}\left[\frac{a^k}{k!}+\frac{a^{k-1}}{(k-1)!}+\frac{a^{k-2}}{(k-2)!}+\cdots+\frac{a}{1!}+\frac{1}{0!}\right]$$

$$Pr(Y=k)=\frac{(1-\rho)}{(b-a)}\{(\frac{e^{-a}a^k-e^{-b}b^k}{k!})+(\frac{e^{-a}a^{k-1}-e^{-b}b^{k-1}}{(k-1)!})+\cdots+\frac{e^{-a}a-e^{-b}b}{1!}+(e^{-a}-e^{-b})\}$$

$$\begin{aligned}Pr(Y=k+1) &= \frac{(1-\rho)}{(b-a)}[(\frac{e^{-a}a^{k+1}-e^{-b}b^{k+1}}{(k-1)!})+(\frac{e^{-a}a^k-e^{-b}b^k}{k!})+\cdots+ \\ &\quad (\frac{e^{-a}a-e^{-b}b}{1})+(e^{-a}-e^{-b})] \\ &= \frac{(1-\rho)}{(b-a)}\{\frac{e^{-a}a^{k+1}-e^{-b}b^{k+1}}{(k+1)!}\}+Pr(Y=k)\end{aligned}$$

The recursive formula becomes

$$Prob(Y=k+1) = \begin{cases}\rho+\frac{(1-\rho)}{b-a}[e^{-a}-e^{-b}], & \text{k=0;} \\ \frac{(1-\rho)}{(b-a)}\{\frac{e^{-a}a^{k+1}-e^{-b}b^{k+1}}{(k+1)!}\}+Pr(Y=k), & \text{k=1, 2, \ldots.}\end{cases} \quad (3)$$

with $Pr(Y=-1)=0$

3 Poisson-Inverse Gaussian Distribution

If the Inverse Gaussian mixing distribution is given by

$$g(\lambda)=(\frac{\phi}{2\pi\lambda^3})^{\frac{1}{2}}\exp\{\frac{-\phi(\lambda-\mu)^2}{2\mu^2\lambda}\}\ \lambda>0,\ \mu>0,\ \phi>0$$

then the recursive formula for Zero Inflated Poisson-Inverse Gaussian distribution becomes

$$\begin{aligned}Pr(Y=k) &= \begin{cases}\int_0^\infty[\rho+(1-\rho)e^{-\lambda}](\frac{\phi}{2\pi\lambda^3})^{\frac{1}{2}}\exp\{\frac{-\phi(\lambda-\mu)^2}{2\mu^2\lambda}\}d\lambda, & \text{k=0;} \\ \int_0^\infty[(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}](\frac{\phi}{2\pi\lambda^3})^{\frac{1}{2}}\exp\{\frac{-\phi(\lambda-\mu)^2}{2\mu^2\lambda}\}d\lambda, & \text{k=1, 2, \ldots.}\end{cases} \\ &= \begin{cases}\rho+(1-\rho)(\frac{\phi}{2\pi})^{\frac{1}{2}}e^{\frac{\phi}{\mu}}\int_0^\infty\lambda^{\frac{-3}{2}}e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}}e^{\frac{\phi}{\mu}}\int_0^\infty\lambda^{k-\frac{3}{2}}e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}}d\lambda, & \text{k=1, 2, \ldots.}\end{cases}\end{aligned}$$

Let

$$I_k=\int_0^\infty\lambda^{k-\frac{3}{2}}e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}}d\lambda$$

Using integration by parts , let

$$u = e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}}$$

and

$$dv = \lambda^{k-\frac{3}{2}} d\lambda$$

then

$$du = \left[-(1+\frac{\phi}{2\mu^2}) + \frac{\phi}{2\lambda^2}\right] e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}} d\lambda$$

and

$$v = \frac{\lambda^{k+1-\frac{3}{2}}}{k+1-\frac{3}{2}}$$

This implies that

$$\begin{aligned} I_k &= -\int_0^\infty \frac{\lambda^{k+1-\frac{3}{2}}}{k+1-\frac{3}{2}} \left[-(1+\frac{\phi}{2\mu^2}) + \frac{\phi}{2\lambda^2}\right] e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}} d\lambda \\ &= \left(1+\frac{\phi}{2\mu^2}\right) \frac{1}{(k+1-\frac{3}{2})} \int_0^\infty \lambda^{k+1-\frac{3}{2}} e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}} d\lambda \\ &\quad - \frac{\phi}{2(k+1-\frac{3}{2})} \int_0^\infty \lambda^{k-1-\frac{3}{2}} e^{-\lambda(1+\frac{\phi}{2\mu^2})-\frac{\phi}{2\lambda}} d\lambda \\ &= \left(1+\frac{\phi}{2\mu^2}\right) \frac{1}{(k+1-\frac{3}{2})} I_{k+1} - \frac{\phi}{2(k+1-\frac{3}{2})} I_{k-1} \end{aligned}$$

This implies that

$$k!Pr(Y=k) = \left(1+\frac{\phi}{2\mu^2}\right) \frac{(k+1)!}{(k+1-\frac{3}{2})} Pr(Y=k+1) - \frac{\phi(k-1)!}{2(k+1-\frac{3}{2})} Pr(Y=k-1)$$

and

$$kPr(Y=k) = (1+\frac{\phi}{2\mu^2}) \frac{k(k+1)}{(k+1-\frac{3}{2})} Pr(Y=k+1) - \frac{\phi}{2k-1} Pr(Y=k-1)$$

Therefore, the recursive formula is

$$\begin{aligned} &(1-\rho)\left[(1+\frac{\phi}{2\mu^2}) \frac{k(k+1)}{(k+1-\frac{3}{2})} Pr(Y=k+1)\right] \\ &= (1-\rho)\left[kPr(Y=k) + \frac{\phi}{2k-1} Pr(Y=k-1)\right] \qquad (4) \end{aligned}$$

with

$$Pr(Y=-1) = 0$$

4 Poisson-Exponential with One parameter

If the distribution for the exponential with one parameter is given by

$$g(\lambda) = \mu\, e^{-\mu\lambda}\ \lambda > 0,\ \mu > 0,$$

then the recursive formula for the Zero Inflated Poisson-Exponential with one parameter distribution becomes

$$Pr(Y = k) \quad = \quad \begin{cases} \rho + (1-\rho)\mu \int_0^\infty e^{-(1+\mu)\lambda} d\lambda, & \text{k=0;} \\ (1-\rho)\frac{\mu}{k!} \int_0^\infty e^{-(1+\mu)\lambda} \lambda^k d\lambda & \text{k=}1, 2, \ldots. \end{cases}$$

Let

$$I_k = \frac{k! Pr(Y = k)}{(1-\rho)} \approx \int_0^\infty \mu\, e^{-(1+\mu)\lambda} \lambda^k d\lambda$$

Using integration by parts,

$$u = e^{-(1+\mu)\lambda}$$

and

$$dv = \lambda^k d\lambda$$

, then

$$du = -(1+\mu) e^{-(1+\mu)\lambda} d\lambda$$

Then

$$\begin{aligned} I_k &= \frac{1}{k+1} \int_0^\infty (1+\mu) e^{-(1+\mu)\lambda} \lambda^{k+1} d\lambda \\ &= \left(\frac{1+\mu}{k+1}\right) I_{k+1} \end{aligned}$$

$$I_{k+1} = \left(\frac{k+1}{1+\mu}\right) I_k$$

It follows that

$$\frac{(k+1)! Pr(Y = k+1)}{(1-\rho)\mu} = \left(\frac{k+1}{1+\mu}\right) \frac{k!}{\mu} Pr(Y = k)$$

Therefore

$$\begin{aligned} Pr(Y = k+1) &= \left(\frac{k+1}{1+\mu}\right) \frac{k!}{(k+1)!} Pr(Y = k) \\ &= \left(\frac{1}{1+\mu}\right) Pr(Y = k) \end{aligned}$$

The recursive formular

$$Pr(Y = k+1) \quad = \quad \begin{cases} \rho + (1-\rho)\frac{\mu}{(1+\mu)}, & \text{k=0;} \\ \left(\frac{1}{1+\mu}\right) Pr(Y = k), & \text{k=}1, 2, \ldots. \end{cases} \tag{5}$$

5 Poisson-Gamma with Two parameters

If the pdf of a Gamma distribution with two parameters is given by

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma\alpha} e^{-\beta\lambda}\lambda^{\alpha-1},\ \lambda > 0.\ \alpha > 0,\ \beta > 0$$

then the recursive formula for Zero Inflated Poisson-Gamma with two parameters becomes

$$Pr(Y = k) = \begin{cases} \rho + (1-\rho)\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-(1+\beta)\lambda}\lambda^{\alpha-1}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-(1+\beta)\lambda}\lambda^{\alpha-1+k}d\lambda, & \text{k=}1, 2, \ldots. \end{cases}$$

Now

$$\frac{k!}{(1-\rho)}\frac{\Gamma\alpha}{\beta^\alpha}P(Y = k) = I_k \approx \int_0^\infty e^{-(1+\beta)\lambda}\lambda^{\alpha-1+k}d\lambda$$

Using integration by parts

$$\begin{aligned} I_k &= \int_0^\infty \frac{1+\beta}{\alpha+k}e^{-(1+\beta)\lambda}\lambda^{\alpha-1+k}d\lambda \\ &= \left(\frac{1+\beta}{\alpha+k}\right) I_{k+1} \end{aligned}$$

Then

$$I_{k+1} = \left(\frac{\alpha+k}{1+\beta}\right) I_k$$

It follows that

$$\frac{(k+1)!}{(1-\rho)}\frac{\Gamma\alpha}{\beta^\alpha}Pr(Y = k+1) = \frac{(\alpha+k)}{(1+\beta)}\frac{k!}{(1-\rho)}\frac{\Gamma\alpha}{\beta^\alpha}Pr(Y = k)$$

Thus

$$Pr(Y = k+1) = \left(\frac{\alpha+k}{(1+\beta)(k+1)}\right) Pr(Y = k)$$

The recursive formula becomes

$$Pr(Y = k+1) = \begin{cases} \rho + (1-\rho)\left(\frac{\beta}{1+\beta}\right)^\alpha, & \text{k=0;} \\ \left(\frac{\alpha+k}{(k+1)(1+\beta)}\right) Pr(Y = k), & \text{k=}1, 2, \ldots. \end{cases} \tag{6}$$

6 Mixing with Poisson - Beta distribution

The Poisson - Beta distribution is

$$g(\lambda) = \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)},\ 0 \le \lambda \le\ 1$$

The recursive formula is derived as follows;

$$Pr(Y=k) = \begin{cases} \rho + (1-\rho)\int_0^1 e^{-\lambda}\frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\int_0^1 e^{-\lambda}\lambda^k\frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}d\lambda, & \text{k=}1,2,\ldots. \end{cases}$$

$$= \begin{cases} \rho + \frac{(1-\rho)}{B(\alpha,\beta)}\int_0^1 e^{-\lambda}\lambda^{\alpha-1}(1-\lambda)^{\beta-1}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!B(\alpha,\beta)}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^{\beta-1}d\lambda, & \text{k=}1,2,\ldots. \end{cases}$$

Now,

$$I_k(\alpha,\beta) = \frac{k!B(\alpha,\beta)}{(1-\rho)}Pr(Y=k) \approx \int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^{\beta-1}d\lambda$$

Let

$$u = e^{-\lambda}\lambda^{\alpha+k-1}$$

and

$$dv = (1-\lambda)^{\beta-1}d\lambda$$

Therefore

$$du = -e^{-\lambda}\lambda^{\alpha+k-1} + (\alpha+k-1)e^{-\lambda}\lambda^{\alpha+k-2}d\lambda$$

and

$$v = -\frac{(1-\lambda)^\beta}{\beta}$$

Therefore

$$\begin{aligned} I_k(\alpha,\beta) &= -\frac{1}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^\beta d\lambda + \frac{(\alpha+k-1)}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-2}(1-\lambda)^\beta d\lambda \\ &= -\frac{1}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)(1-\lambda)^{\beta-1}d\lambda + \\ &\quad \frac{(\alpha+k-1)}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-2}(1-\lambda)(1-\lambda)^{\beta-1}d\lambda \\ &= -\frac{1}{\beta}\{\int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^{\beta-1}d\lambda - \int_0^1 e^{-\lambda}\lambda^{\alpha+k}(1-\lambda)^{\beta-1}d\lambda\} \\ &\quad +\{-\frac{\alpha+k-1}{\beta}\int_0^1 e^{-\lambda}\lambda^{\alpha+k-2}(1-\lambda)^{\beta-1}d\lambda - \int_0^1 e^{-\lambda}\lambda^{\alpha+k-1}(1-\lambda)^{\beta-1}d\lambda\} \\ &= -\frac{1}{\beta}\{I_k(\alpha,\,\beta) - I_{k+1}(\alpha,\,\beta)\} + \frac{(\alpha+k-1)}{\beta}\{I_{k-1}(\alpha,\,\beta) - I_k(\alpha,\,\beta)\} \end{aligned}$$

Thus

$$\begin{aligned} I_{k+1}(\alpha,\,\beta) &= \beta\, I_k(\alpha,\,\beta) + I_k(\alpha,\,\beta) + (\alpha+k-1)I_k(\alpha,\,\beta) - (\alpha+k-1)I_{k-1}(\alpha,\,\beta) \\ &= (\alpha+\beta+k)I_k(\alpha,\,\beta) - (\alpha+k-1)I_{k-1}(\alpha,\,\beta) \end{aligned}$$

This implies that

$$(k+1)!Pr(Y=k+1) = (\alpha+\beta+k)k!Pr(Y=k) - (\alpha+k-1)(k-1)!Pr(Y=k-1) \tag{7}$$

Hence the recursive formular is

$$k(k+1)Pr(Y=k+1) = (\alpha+\beta+k)kPr(Y=k) - (\alpha+k-1)Pr(Y=k-1) \tag{8}$$

with

$$Pr(Y=-1)=0$$

7 Mixing with Inverted - Beta distribution

The mixing distribution is

$$g(\lambda) = \frac{\lambda^{\alpha-1}}{B(\alpha,\,\beta)(1+\lambda)^{\alpha+\beta}},\ \lambda>0,\ \alpha>0,\ \beta>0$$

The mixed distribution is

$$Pr(Y=k) \quad = \quad \begin{cases} \rho + (1-\rho)\int_0^\infty e^{-\lambda}\frac{\lambda^{\alpha-1}}{B(\alpha,\beta)(1+\lambda)^{\alpha+\beta}}d\lambda, & \text{k=0;} \\ \frac{(1-\rho)}{k!}\int_0^\infty e^{-\lambda}\lambda^k\frac{\lambda^{\alpha-1}}{B(\alpha,\beta)(1+\lambda)^{\alpha+\beta}}d\lambda, & \text{k=1}, 2, \ldots. \end{cases}$$

$$\frac{k!B(\alpha,\,\beta)}{(1-\rho)}Pr(Y=k) = \int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-1}}{(1+\lambda)^{\alpha+\beta}}d\lambda \approx\ I_x$$

Let

$$u = e^{-\lambda}\lambda^{\alpha+k-1}$$

and

$$dv = \frac{d\lambda}{(1+\lambda)^{\alpha+\beta}}$$

then

$$du = -e^{-\lambda}\lambda^{\alpha+k-1} + e^{-\lambda}(\alpha+k-1)\lambda^{\alpha+k-2}$$

and

$$v = -\frac{(1+\lambda)^{-(\alpha+\beta-1)}}{(\alpha+\beta-1)}$$

$$\begin{aligned} I_k \quad &= \quad \frac{1}{(\alpha+\beta-1)}\int_0^\infty (1+\lambda)^{-(\alpha+\beta-1)}\{e^{-\lambda}(\alpha+k-1)\lambda^{\alpha+k-2} - e^{-\lambda}\lambda^{\alpha+k-1}\}d\lambda \\ &= \quad \frac{1}{(\alpha+\beta-1)}\{\int_0^\infty \frac{e^{-\lambda}(\alpha+k-1)\lambda^{\alpha+k-2}(1+\lambda)}{(1+\lambda)(1+\lambda)^{-(\alpha+\beta-1)}}d\lambda - \int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-1}(1+\lambda)}{(1+\lambda)(1+\lambda)^{-(\alpha+\beta-1)}}d\lambda\} \\ &= \quad \frac{(\alpha+k-1)}{(\alpha+\beta-1)}\{\int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-2}}{(1+\lambda)^{-(\alpha+\beta)}}d\lambda + \int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-1}}{(1+\lambda)^{-(\alpha+\beta)}}d\lambda\} - \\ &\quad\quad \frac{1}{(\alpha+\beta-1)}\{\int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k-1}}{(1+\lambda)^{-(\alpha+\beta)}}d\lambda + \int_0^\infty \frac{e^{-\lambda}\lambda^{\alpha+k}}{(1+\lambda)^{-(\alpha+\beta)}}d\lambda\} \\ &= \quad \frac{(\alpha+k-1)}{(\alpha+\beta-1)}\{I_{k-1}+I_k\} - \frac{1}{(\alpha+\beta-1)}\{I_k+I_{k+1}\} \end{aligned}$$

Therefore

$$(\alpha+\beta-1)I_k = (\alpha+k-1)(I_{k-1}+I_k) - (I_k+I_{k+1})$$

which implies that

$$I_{k+1} = (\alpha+k-1)I_k + (k-\beta-1)I_{k-1}$$

Thus

$$\frac{(k+1)!B(\alpha,\,\beta)}{(1-\rho)}Pr(Y=k+1) = \frac{(k-\beta-1)k!B(\alpha,\,\beta)}{(1-\rho)}Pr(Y=k)$$
$$+\frac{(\alpha+k-1)(k-1)!B(\alpha,\,\beta)}{(1-\rho)}Pr(Y=k-1) \qquad (9)$$

which when simplified gives the recursive formular a Zero Inflated Poisson-Inverted Beta distribution as

$$k(k+1)Pr(Y=k+1) = k(k-\beta-1)Pr(Y=k) + (\alpha+k-1)Pr(Y=k-1),$$

with

$$Pr(Y=-1)=0$$

8 Conclusion

From the above continuous prior distributions, it can be clearly seen that the recursive relations can be derived for numerous mixture distributions. This is made possible by the fact that there are no restrictions imposed during integration. It is known that the distributions do not exist when the variable $k<0$. We restricted ourselves to continuous mixing distribution even though, discrete or countable mixtures where we have discrete prior distributions could be of interest to a researcher, thus, research can be carried out on this.

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