



The Co-operative University of Kenya

END OF SEMESTER EXAMINATIONS

EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE IN ACTUARIAL SCIENCE, APPLIED STATISTICS, AND STATISTICS AND INFORMATION TECHNOLOGY

COURSE CODE: **BSTA 2206**

UNIT TITLE: **PROBABILITY AND STATISTICS III**

DATE: **APRIL, 2023**

Time: **2 Hours**

INSTRUCTIONS

1. Answer question **ONE** and any other **TWO (2)** questions
3. Scientific Calculators and non-programmable calculators may be used
4. Use the provided statistical tables where applicable

1. (a) Define the terms
 - i. Discrete bi-variate random variables (1 marks)
 - ii. Joint probability density function (1 marks)
 - iii. Marginal probability density function (1 marks)
- (b) Suppose X and Y are discrete random variables with joint probability distributions given by

$$f(x, y) = \begin{cases} \frac{1}{30}(x + y) & , x = 0, 1, 2, 3; y = 0, 1, 2 \\ 0 & , \text{elsewhere} \end{cases}$$
 - i. Verify that $f(x, y)$ is a probability distribution (3 marks)
 - ii. Work out $P(X \leq 2)$ (3 marks)
- (c) Let X and Y be jointly distributed random variables if the joint probability function

$$f(x, y) = \begin{cases} p^{x+y}(1-p)^{2-x-y} & , x = 0, 1; y = 0, 1 \\ 0 & , \text{elsewhere} \end{cases}$$

Compute

 - i) $E(X)$ (3 marks)
 - ii) $E(X + Y)$ (4 marks)
 - iii) $E(XY)$ (3 marks)
- (d) Let X_1 and X_2 be independent random variable having the probability distribution

$$f(x) = \begin{cases} e^{-x} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

Define $Y_1 = x_1 + x_2$ and $Y_2 = \frac{x_1}{x_1 + x_2}$

 - i. Find the joint PDF of Y_1 and Y_2 using the transformation technique. (7 marks)
 - ii. Show that Y_1 and Y_2 are independent (4 marks)
2. (a) Consider bivariate function defined as

$$f(x, y) = \begin{cases} k(6 - x - y) & , 0 < x < 2, 2 < y < 4 \\ 0 & , \text{elsewhere} \end{cases}$$

- i. Determine the constant k so that the function $f(x, y)$ is a joint pdf of a random variable x and y (3 marks)
- ii. Hence compute the probabilities
 - a. $P(x < 1, y < 3)$ (3 marks)
 - b. $P(x + y < 3)$ (4 marks)
- (b) Consider two random samples, X_1 and X_2 of sizes 10 and 20 with sample variances given as 0.0003 and 0.0001 respectively. Assuming that the populations, from which the samples have been drawn, are normal, determine whether the variance of the first population is significantly greater than the second one. Take $\alpha = 0.05$ (6 marks)
- (c) Define the standard versions of X and Y as

$$U = \frac{X - EX}{\sigma_X}, V = \frac{Y - EY}{\sigma_Y}$$

Show that

$$\rho_{XY} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

(4 marks)

3. (a) Suppose that X and Y are two discrete random variables with the joint probability functions given by $f(x, y) = \begin{cases} \frac{1}{54}(x + y) & , x = 1, 2, 3; y = 1, 2, 3, 4 \\ 0 & , \text{elsewhere} \end{cases}$

Determine the marginal probability function of x and y (7 marks)

- (b) A machine engine part produced by a company is claimed to have diameter variance no larger than 0.0002 (diameter measured in inches). A random sample of 10 parts gave a sample variance of 0.0003. Test, at the 5% level, $H_0 : \sigma^2 = 0.0002$ against $H_1 : \sigma^2 > 0.0002$. (6 marks)

- (c) The joint PDF of X_1 and X_2 is given by $f(x_1, x_2) = \begin{cases} 3x_1 & , 0 \leq x_2 \leq x_1 \leq 1 \\ 0 & , \text{elsewhere} \end{cases}$

Find

- i. $Cov(X_1, X_2)$ (5 marks)
 - ii. Correlation coefficient ρ_0 (2 marks)
4. (a) An element has two electrons. Each of the electrons can occupy one of the four orbits of this element. Let the random variables x and y denote the orbit number occupied by these two electrons. The joint probability distribution is given as

$f(x, y)$	$x = 1$	$x = 2$	$x = 3$	$x = 4$
$y = 1$	0	$\frac{1}{16}$	$\frac{1}{16}$	0
$y = 2$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$
$y = 3$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$
$y = 4$	0	$\frac{1}{16}$	$\frac{1}{16}$	0

- i. Are these random variables independent? Prove your answer (3 marks)

- ii. You have a vacuum chamber containing 6 atoms of this element type. If the sum of the orbit numbers for any atom is greater than 5, there is a 80% probability that this atom will release a photon. If the sum of the orbit numbers for any atom is less than or equal to 6, no photon will be released. What is the probability that exactly 3 photons will be released from the atom in the vacuum chamber? **(5 marks)**
- iii. Given that $x \sim \chi^2(5)$. Determine the probability that x is between 1.145 and 12.83 **(3 marks)**
- iv. If $x \sim f(9, 10)$ determine
- A. $P(x \geq 3.02)$ **(3 marks)**
- B. The mean and variance of x **(3 marks)**
- v. Suppose x_1, x_2, \dots, x_{10} are observation from a random sample of size 10 from a distribution with density $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} - \infty < x < \infty$ Determine the moment generating function of the sample **(3 marks)**
5. (a) Let U_1 and U_2 be independent random variables where U_1 has a chi-square distribution with v_1 degrees of freedom and U_2 has a chi-square distribution with v_2 degrees of freedom. Derive the probability distribution function of the random variable

$$F = \frac{U_1/V_1}{U_2/V_2}$$

(11 marks)

- (b) Let X_1, X_2 and X_3 be independent random variables normally distributed with mean 5 and variance $\sigma^2 = 4$. Let $F = \frac{2}{3} \left[\frac{(x_1-5)^2 + (x_2-5)^2 + (x_3-5)^2}{(x_2-5)^2 + (x_3-5)^2} \right]$
- i. Derive the distribution of F **(5 marks)**
- ii. Find c if $P(F \geq c) = 0.05$ **(4 marks)**

Chi-square Distribution Table

d.f.	.995	.99	.975	.95	.9	.1	.05	.025	.01
1	0.00	0.00	0.00	0.00	0.02	2.71	3.84	5.02	6.63
2	0.01	0.02	0.05	0.10	0.21	4.61	5.99	7.38	9.21
3	0.07	0.11	0.22	0.35	0.58	6.25	7.81	9.35	11.34
4	0.21	0.30	0.48	0.71	1.06	7.78	9.49	11.14	13.28
5	0.41	0.55	0.83	1.15	1.61	9.24	11.07	12.83	15.09
6	0.68	0.87	1.24	1.64	2.20	10.64	12.59	14.45	16.81
7	0.99	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.72
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89
32	15.13	16.36	18.29	20.07	22.27	42.58	46.19	49.48	53.49
34	16.50	17.79	19.81	21.66	23.95	44.90	48.60	51.97	56.06
38	19.29	20.69	22.88	24.88	27.34	49.51	53.38	56.90	61.16
42	22.14	23.65	26.00	28.14	30.77	54.09	58.12	61.78	66.21
46	25.04	26.66	29.16	31.44	34.22	58.64	62.83	66.62	71.20
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15
55	31.73	33.57	36.40	38.96	42.06	68.80	73.31	77.38	82.29
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38
65	39.38	41.44	44.60	47.45	50.88	79.97	84.82	89.18	94.42
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.43
75	47.21	49.48	52.94	56.05	59.79	91.06	96.22	100.84	106.39
80	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33
85	55.17	57.63	61.39	64.75	68.78	102.08	107.52	112.39	118.24
90	59.20	61.75	65.65	69.13	73.29	107.57	113.15	118.14	124.12
95	63.25	65.90	69.92	73.52	77.82	113.04	118.75	123.86	129.97
100	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81